

ON THE ANALOGS OF EULER NUMBERS AND POLYNOMIALS ASSOCIATED WITH p -ADIC q -INTEGRAL ON \mathbb{Z}_p AT $q = -1$

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ABSTRACT. The purpose of this paper is to construct of λ -Euler numbers and polynomials by using fermionic expression of p -adic q -integral at $q = -1$. From these λ -Euler polynomials, we derive the harmonic sums of higher order. Finally, we investigate several interesting properties and relationships involving the classical as well as the generalized Euler numbers and polynomials. As an application, we will treat p -adic invariant integrals on \mathbb{Z}_p involving trigonometric functions.

§1. Introduction

Let p be a fixed odd prime, and let \mathbb{C}_p be the p -adic completion of the algebraic closure of \mathbb{Q}_p . For a fixed positive integer d with $(p, d) = 1$, set

$$\begin{aligned} X &= X_d = \varprojlim_{\mathbb{N}} \mathbb{Z}/dp^N \mathbb{Z}, \\ X_1 &= \mathbb{Z}_p, \\ X^* &= \bigcup_{0 < a < dp} (a + dp\mathbb{Z}_p), \\ a + dp^N \mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^n}\}, \end{aligned}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$ [1].

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The p -adic absolute value in \mathbb{C}_p is normalized in such way that $|p|_p = 1/p$. Let $U_1 \subset \mathbb{C}_p$ denote the open unit disc about 1 and $U_d = \{u \in \mathbb{C}_p \mid |u^d - 1|_p < 1\}$ the union of the open unit discs around d -th root of unity. Let $U^m = U_d \times U_1^{m-1}$. When one talks of q -extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or a p -adic number $q \in \mathbb{C}_p$.

If $q \in \mathbb{C}$, then we normally assume

$$|q - 1|_p < p^{-1/p-1} \quad \text{so that} \quad q^x = \exp(x \log q) \quad \text{for } |x|_p \leq 1.$$

Throughout this paper, we use the below notation :

$$[x]_q = [x : q] = \frac{1 - q^x}{1 - q} = 1 + q + q^2 + \cdots + q^{x-1}.$$

For $f \in C^{(1)}(\mathbb{Z}_p)$ (= the set of strictly differentiable function on \mathbb{Z}_p), let us start with the expressions

$$\frac{1}{[p^N]_q} \sum_{0 \leq x < p^N} q^x f(x) = \sum_{0 \leq x < p^N} f(x) \mu_q(x + p^N \mathbb{Z}_p), \quad \text{cf. [11],}$$

representing the q -analogue of the Riemann sums for f . The integral of f on \mathbb{Z}_p will be defined as the limit ($N \rightarrow \infty$) of these sums, when it exists. The p -adic q -integral of a function $f \in C^{(1)}(\mathbb{Z}_p)$ is defined by author as follows:

$$\begin{aligned} I_q(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_q(x) \\ &= \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{0 \leq x < p^N} f(x) q^x, \quad \text{see [7,8,9,10].} \end{aligned} \tag{1}$$

For $f \in C^{(1)}(\mathbb{Z}_p)$, it is easy to see that

$$\left| \int_{\mathbb{Z}_p} f(x) d\mu_q(x) \right|_p \leq p \|f\|_1, \quad \text{cf. [2,3,4],}$$

where $\|f\|_1 = \sup \left\{ |f(0)|_p, \sup_{x \neq y} \left| \frac{f(x) - f(y)}{x - y} \right|_p \right\}$.

If $f_n \rightarrow f$ in $C^{(1)}(\mathbb{Z}_p)$, namely $\|f_n - f\|_1 \rightarrow 0$, then

$$\int_{\mathbb{Z}_p} f_n(x) d\mu_q(x) \rightarrow \int_{\mathbb{Z}_p} f(x) d\mu_q(x), \quad \text{see [6,7,12].}$$

The q -analogue of the binomial coefficient is known as

$$\begin{bmatrix} x \\ n \end{bmatrix}_q = \frac{[x]_q [x-1]_q \cdots [x-n+1]_q}{[n]_q!},$$

where $[n]_q! = \prod_{i=1}^n [i]_q$, cf. [11].

From this, we see that

$$\int_{\mathbb{Z}_p} \begin{bmatrix} x \\ n \end{bmatrix}_q d\mu_q(x) = \frac{(-1)^n}{[n+1]_q} q^{n+1-\binom{n+1}{2}}, \quad (\text{see [11]}). \quad (2)$$

For $q \in [0, 1]$ certain q -deformed bosonic operators may be introduced which generalize the undeformed bosonic ones (corresponding $q = 1$), see [13,14,15].

The Eq. (2) is useful to study q -deformed Stirling numbers $S_q(n, k)$ in the version introduced by Milne [14]. From (1), we derive the usual “bosonic” p -adic q -integral as the limit $q \rightarrow 1$, i.e., $I_{q=1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{q=1}(x)$ [4,5,6]. It is possible to consider the case $q \in (-1, 0)$ corresponding to q -deformed fermionic certain and annihilation operators and the literature given therein [16,22].

The expression (and the recursion relation) for the $I_q(f)$ remains the same, so it is tempting to consider the limit $q \rightarrow -1$. This limit will be called “fermionic” since the corresponding certain and annihilation operators are those of an undeformed fermion. It is well known that the familiar Bernoulli polynomials $B_n(x)$ are defined by means of the following generating function:

$$F(x, t) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \text{cf. [6,19,20]}, \quad (3)$$

we note that, by substituting $x = 0$ into (3), $B_n(0) = B_n$ are the n -th Bernoulli numbers. By using “bosonic” expression of p -adic q -integral, one decade ago, we defined λ -extension of these classical Bernoulli polynomials and proved the properties of analogs to those satisfied by B_n and $B_n(x)$ [3,4,6,19]. That is,

$$\begin{aligned} \lim_{q \rightarrow 1} \int_{\mathbb{Z}_p} f(x) \lambda^x d\mu_q(x) &= \frac{\log \lambda + t}{\lambda e^t - 1} \\ &= \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}. \end{aligned} \quad (4)$$

Let C_{p^n} be the cyclic group consisting of all p^n -th roots of unity in \mathbb{C}_p for any $n \geq 0$ and T_p be the direct limit of C_{p^n} with respect to the natural morphisms, hence T_p is the union of all C_{p^n} with discrete topology.

In the special $\lambda \in T_p$, we note that

$$\begin{aligned} \lim_{q \rightarrow 1} \int_{\mathbb{Z}_p} \lambda^x e^{tx} d\mu_q(x) &= \frac{t}{\lambda e^t - 1} \\ &= \frac{\log \lambda + t}{\lambda e^t - 1} \\ &= \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}, \end{aligned}$$

where $B_n(\lambda)$ are n -th λ -Bernoulli numbers [2,6]. Now, we consider the case $q \in (-1, 0)$ corresponding to q -deformed fermionic certain and annihilation operators and the literature given therein. The expression for the $I_q(f)$ remains same, so it is tempting to consider the limit $q \rightarrow -1$. That is,

$$\begin{aligned} I_{q=-1}(f) &= \lim_{q \rightarrow -1} I_q(f) \\ &= \int_{\mathbb{Z}_p} f(x) d\mu_{q=-1}(x), \text{ see [7,11]}. \end{aligned}$$

Thus, we note that

$$\begin{aligned} \int_{\mathbb{Z}_p} e^{tx} \lambda^x d\mu_{q=-1}(x) &= \frac{2}{\lambda e^t + 1} \\ &= \sum_{n=0}^{\infty} E_n(\lambda) \frac{t^n}{n!}, \end{aligned} \tag{5}$$

where $E_n(\lambda)$ are called λ -Euler numbers.

These numbers are classical and important in number theory. In this paper we construct λ -Euler polynomials and numbers by using “fermionic” expression of p -adic q -integral. From these λ -Euler polynomials, we derive the harmonic sums of higher order. Finally we investigate several interesting properties and relationships involving the classical as well as the generalized Euler numbers and polynomials. As an application, we will treat p -adic invariant integrals on \mathbb{Z}_p involving trigonometric functions.

§2. λ -Euler numbers and polynomials associated with fermionic expression of p -adic q -integral

For $f \in C^{(1)}(\mathbb{Z}_p)$, we consider “fermionic” expression for the p -adic q -integral on \mathbb{Z}_p as follows:

$$\begin{aligned} I_{-1}(f) &= I_{q=-1}(f) = \lim_{q \rightarrow -1} I_q(f) \\ &= \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x). \end{aligned} \quad (6)$$

Let $f_1(x)$ be translation with $f_1(x) = f(x+1)$. Then we see that

$$\begin{aligned} I_{-1}(f_1) &= - \lim_{n \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x + 2f(0) \\ &= -I_{-1}(f) + 2f(0). \end{aligned}$$

Therefore we obtain the following:

Lemma 1. *For $f \in C^{(1)}(\mathbb{Z}_p)$, we have*

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0). \quad (7)$$

From (7), we can easily derive the below theorem.

Theorem 2. *For $f \in C^{(1)}(\mathbb{Z}_p)$, $n \in \mathbb{N}$, we have*

$$I_{-1}(f_n) + (-1)^n I_{-1}(f) = 2 \sum_{x=0}^{n-1} (-1)^{n-1-x} f(x),$$

where $f_n(x) = f(x+n)$.

By Theorem 2, we can consider λ -Euler polynomials. If we take $f(x) = \lambda^x e^{xt}$ ($\lambda \in \mathbb{Z}_p$), then we have

$$\frac{2}{\lambda e^t + 1} e^{xt} = \int_{\mathbb{Z}_p} e^{(x+y)t} \lambda^y d\mu_{-1}(y). \quad (8)$$

Now we define λ -Euler polynomials as follows:

$$F_\lambda(t, x) = \frac{2}{\lambda e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(\lambda : x) \frac{t^n}{n!}, \quad (9)$$

where $E_n(\lambda : x)$ are called n -th λ -Euler polynomials. We also define, by substituting $x = 0$ into (9), $E_n(\lambda : 0) = E_n(\lambda)$ n -th λ -Euler numbers. From (8) and (9), we derive the With's formula for $E_n(\lambda; x)$ as

$$\int_{\mathbb{Z}_p} \lambda^y (x + y)^n d\mu_{-1}(y) = E_n(\lambda : x). \quad (9-1)$$

Note that

$$\begin{aligned} E_n(\lambda : x) &= \sum_{l=0}^n \binom{n}{l} E_l(\lambda) x^{n-l} \\ &= d^n \sum_{a=0}^{d-1} (-1)^a \lambda^a E_n(\lambda^d : \frac{a+x}{d}), \end{aligned} \quad (10)$$

where d is positive integer. Because

$$\int_{\mathbb{Z}_p} (x + y)^n \lambda^y d\mu_{-1}(y) = d^n \sum_{a=0}^{d-1} (-1)^a \lambda^a \int_{\mathbb{Z}_p} (\frac{a+x}{d} + y)^n \lambda^{dy} d\mu_{-1}(y).$$

Remark. In [2,3,4,5,6,19], it was known that

$$\begin{aligned} \lim_{q \rightarrow 1} \int_{\mathbb{Z}_p} \lambda^y e^{(x+y)t} d\mu_q(y) &= \frac{t + \log \lambda}{\lambda e^t - 1} e^{xt} \\ &= \sum_{m=0}^{\infty} B_m(\lambda; x) \frac{t^m}{m!}, \end{aligned} \quad (11)$$

where $B_m(\lambda; x)$ are called m -th λ -Bernoulli polynomials.

In view point of (11), we considered λ -Euler numbers and polynomials.

Let χ be the Dirichlet's character with conductor $d(= \text{odd}) \in \mathbb{N}$ and let us take $f(x) = \chi(x) e^{tx} \lambda^x$.

From Theorem 2, we derive the below formula:

$$\int_{\chi} \lambda^x e^{tx} \chi(x) d\mu_{-1}(x) = \frac{2 \sum_{a=0}^{d-1} e^{ta} (-1)^a \chi(a) \lambda^a}{\lambda^d e^{dt} + 1}. \quad (12)$$

By (12), we also consider the generalized λ -Euler numbers attached to χ as follows:

$$\begin{aligned} F_{\lambda, \chi}(t) &= \frac{2 \sum_{a=0}^{d-1} e^{ta} (-1)^a \chi(a) \lambda^a}{\lambda^d e^{dt} + 1} \\ &= \sum_{n=0}^{\infty} E_{n, \chi}(\lambda) \frac{t^n}{n!}. \end{aligned} \quad (13)$$

From (12) and (13), we derive

$$\int_{\chi} \lambda^x x^n \chi(x) d\mu_{-1}(x) = E_{n,\chi}(\lambda), \quad n \geq 0. \quad (14)$$

It is easy to check that

$$\int_{\chi} x^n \chi(x) d\mu_{-1}(x) = d^n \sum_{a=0}^{d-1} (-1)^a \lambda^a \chi(a) \int_{\mathbb{Z}_p} \lambda^{xd} \left(\frac{a}{d} + x \right)^n d\mu_{-1}(x). \quad (15)$$

From (9-1), (13) and (14), we derive

$$E_{n,\chi}(\lambda) = d^n \sum_{a=0}^{d-1} (-1)^a \lambda^a \chi(a) E_n \left(\lambda^d : \frac{a}{d} \right). \quad (16)$$

§3. λ -zeta function associated with λ -Euler numbers

In this section, we assume that $\lambda (\neq -1) \in \mathbb{C}$ with $|\lambda| < 1$. By (9), we see that

$$F_{\lambda}(t, x) = \frac{2}{\lambda e^t + 1} e^{xt} = 2 \sum_{n=0}^{\infty} (-1)^n \lambda^n e^{(n+x)t}. \quad (17)$$

From (9) and (17), we note that

$$E_k(\lambda : x) = \left. \frac{d^k}{dt^k} F_{\lambda}(t, x) \right|_{t=0} = 2 \sum_{n=0}^{\infty} (-1)^n \lambda^n (n+x)^k.$$

Therefore we obtain the following:

Theorem 3. *For $k \in \mathbb{N}$, we have*

$$E_k(\lambda : x) = 2 \sum_{n=0}^{\infty} (-1)^n \lambda^n (n+x)^k.$$

Definition 4. For $s \in \mathbb{C}$, we define λ -zeta function of Hurwitz's type as follows:

$$\zeta_\lambda(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{(n+x)^s}.$$

Note that $\zeta_\lambda(s, x)$ is analytic continuation in whole complex plane. Let n be the even positive integer. Then we have

$$\begin{aligned} \frac{2(1 - \lambda^n e^{nt})}{\lambda e^t + 1} &= 2 \sum_{l=0}^{n-1} (-1)^l \lambda^l e^{lt} \\ &= \sum_{m=0}^{\infty} \left(2 \sum_{l=0}^{n-1} (-1)^l \lambda^l l^m \right) \frac{t^m}{m!}. \end{aligned} \quad (18)$$

By (9) and (18), we see that

$$\sum_{m=0}^{\infty} (E_m(\lambda) - \lambda^n E_m(\lambda : n)) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left(2 \sum_{l=0}^{n-1} (-1)^l \lambda^l l^m \right) \frac{t^m}{m!}. \quad (19)$$

By comparing the coefficients on the both sides in (19), we note that

$$E_m(\lambda) - \lambda^n E_m(\lambda : n) = 2 \sum_{l=0}^{n-1} (-1)^l \lambda^l l^m. \quad (20)$$

From (10) and (20), we derive

$$2 \sum_{l=0}^{n-1} (-1)^{l-1} \lambda^l l^m = \lambda^n \sum_{l=0}^{m-1} \binom{m}{l} E_l(\lambda) n^{m-l} + (\lambda^n - 1) E_m(\lambda). \quad (21)$$

Therefore we obtain the following:

Theorem 5. *Let n be the positive even integer. Then we have*

$$2 \sum_{l=0}^{n-1} (-1)^{l-1} \lambda^l l^m = \lambda^n \sum_{l=0}^{m-1} \binom{m}{l} E_l(\lambda) n^{m-l} + (\lambda^n - 1) E_m(\lambda).$$

Remark.

$$E_0(\lambda) = \frac{2}{\lambda + 1}, \quad E_1(\lambda) = -\frac{2\lambda}{(\lambda + 1)^2}, \quad E_2(\lambda) = \frac{4\lambda^2 - 2\lambda + 2}{(\lambda + 1)^3}, \dots \quad (22)$$

Corollary 6. *Let k be positive integer. Then we see that*

$$\zeta_\lambda(-k, x) = E_k(\lambda : x).$$

Let χ be the primitive Dirichlet's character with conductor $d(=\text{odd}) \in \mathbb{N}$. From (13), we derive

$$\begin{aligned} F_{\lambda, \chi}(t) &= \frac{2 \sum_{a=0}^{d-1} e^{ta} (-1)^a \lambda^a \chi(a)}{\lambda^d e^{dt} + 1} \\ &= 2 \sum_{n=0}^{\infty} (-1)^n \chi(n) \lambda^n e^{nt}. \end{aligned} \tag{23}$$

By (13) and (23), we easily see that

$$\begin{aligned} E_{k, \chi}(\lambda) &= \left. \frac{d^k}{dt^k} F_{\lambda, \chi}(t) \right|_{t=0} \\ &= 2 \sum_{n=1}^{\infty} \chi(n) \lambda^n n^k. \end{aligned}$$

Therefore we obtain the following:

Theorem 7. *Let χ be the primitive Dirichlet's character with conductor $d(=\text{odd}) \in \mathbb{N}$. Then, we have*

$$E_{k, \chi}(\lambda) = 2 \sum_{n=1}^{\infty} (-1)^n \chi(n) n^k \lambda^n.$$

Definition 8. For $s \in \mathbb{C}$, we define $\lambda - l$ -function as follows:

$$l_\lambda(s, \chi) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n \chi(n) \lambda^n}{n^s}. \tag{24}$$

Note that $l_\lambda(s, \chi)$ is also analytic continuation in whole complex plane.

Corollary 9. *Let $k \in \mathbb{N}$. Then we have*

$$l_\lambda(-k, \chi) = E_{k, \chi}(\lambda).$$

Let s be a complex variable, and let a and $F(=\text{odd})$ be integer with $0 < a < F$. We consider the below harmonic sum (partial λ -zeta function):

$$\begin{aligned}
H_\lambda(s, a|F) &= \sum_{\substack{m \equiv a(F) \\ m > 0}} \frac{(-1)^m \lambda^m}{m^s} \\
&= \sum_{m=0}^{\infty} \frac{\lambda^{nF+a} (-1)^{nF+a}}{(a + nF)^s} \\
&= \lambda^a (-1)^a \sum_{n=0}^{\infty} \frac{(\lambda^F)^n (-1)^n}{(a + nF)^s} \\
&= \frac{\lambda^a (-1)^a}{2} F^{-s} \zeta_{\lambda^F}(s, \frac{a}{F}).
\end{aligned}$$

Note that

$$H_\lambda(-k, a|F) = \frac{(-1)^a \lambda^a F^k}{2} E_k(\lambda^F : \frac{a}{F}). \quad (25)$$

From (24) and (25), we derive

$$l_\lambda(s, \chi) = 2 \sum_{a=1}^F \chi(a) H_\lambda(s, a|F).$$

The harmonic sum $H_\lambda(s, a|F)$ will be called partial λ -zeta function which interpolates λ -Euler polynomials at negative integers [21]. The values of $l_\lambda(s, \chi)$ at negative integers are algebraic, hence may be regarded as lying in an extension of \mathbb{Q}_p . We therefore look for a p -adic function which agrees with $l_\lambda(s, \chi)$ at the negative integers in Section 4.

§4. p -adic harmonic sums of higher order associated with p -adic $\lambda - l$ -function

Let $w(x)$ be the Teichmüller character and let $\langle x \rangle = \frac{x}{w(x)}$. When $F(=\text{odd})$ is a multiple of p and $(a, p) = 1$, we define p -adic partial λ -zeta function as follows :

$$H_{\lambda, p}(s, a|F) = \frac{(-1)^a \lambda^a}{2} \langle a \rangle^{-s} \sum_{j=0}^{\infty} \binom{-s}{j} \left(\frac{F}{a} \right)^j E_j(\lambda^F),$$

for $s \in \mathbb{Z}_p$. Thus, we note that

$$\begin{aligned}
H_{\lambda,p}(-n, a|F) &= \frac{(-1)^a \lambda^a}{2} < a >^n \sum_{j=0}^n \binom{n}{j} \left(\frac{F}{a}\right)^j E_j(\lambda^F) \\
&= F^n \frac{(-1)^a \lambda^a}{2} w^{-n}(a) \sum_{j=0}^n \binom{n}{j} \left(\frac{a}{F}\right)^{n-j} E_j(\lambda^F) \\
&= \frac{(-1)^a}{2} F^n w^{-n}(a) E_n(\lambda^F : \frac{a}{F}) \\
&= w^{-n}(a) H_{\lambda}(-n, a|F),
\end{aligned}$$

where n is positive integer.

Now we consider p -adic interpolating function for λ -Euler numbers as follows:

$$l_{\lambda,p}(s, \chi) = 2 \sum_{\substack{a=1 \\ (a,p)=1}}^F \chi(a) H_{\lambda,p}(s, a|F),$$

for $s \in \mathbb{Z}_p$. Let n be natural numbers. Then we have

$$\begin{aligned}
l_{\lambda,p}(-n, \chi) &= 2 \sum_{\substack{a=1 \\ (a,p)=1}}^F \chi(a) H_{\lambda,p}(-n, a|F) \\
&= E_{n, \chi w^{-n}}(\lambda) - p^n \chi w^{-n}(p) E_{n, \chi w^{-n}}(\lambda^p).
\end{aligned}$$

In fact, we see that

$$l_{\lambda,p}(s, \chi) = \sum_{a=1}^F (-1)^a \lambda^a \chi(a) \sum_{j=0}^{\infty} \binom{-s}{j} \left(\frac{F}{a}\right)^j E_j(\lambda),$$

for $s \in \mathbb{Z}_p$. This is a p -adic analytic function and has the following properties for $\chi = w^t$:

$$l_{\lambda,p}(-n, w^t) = E_n(x) - p^n E_n(\lambda^p), \quad (26)$$

where $n \equiv t \pmod{p-1}$, $l_{\lambda,p}(s, w^t) \in \mathbb{Z}_p$ for all $s \in \mathbb{Z}_p$ when $t \equiv 0 \pmod{p-1}$.

If $t \equiv 0 \pmod{p-1}$, then $l_{\lambda,p}(s, w^t) \equiv l_{\lambda,p}(s_2, w^t) \pmod{p}$ for all $s, s_2 \in \mathbb{Z}_p$.

It is easy to see that

$$\frac{1}{r+k-1} \binom{-r}{k} \binom{1-r-k}{j} = \frac{-1}{j+k} \binom{-r}{k+j-1} \binom{k+j}{j},$$

for all positive integers r, j, k with $j, k \geq 0$, $j + k > 0$, and $r \neq 1 - k$.

Thus, we note that

$$\frac{r}{r+k} \binom{-r-1}{k} \binom{-r-k}{j} = \binom{-r}{k+j} \binom{k+j}{j}.$$

For $F(=\text{odd}) \in \mathbb{N}$, let n be positive even integer then we have

$$\begin{aligned} \sum_{l=0}^{n-1} \frac{\lambda^{Fl+a} (-1)^{Fl+a}}{(Fl+a)^r} &= \sum_{l=0}^{n-1} (-1)^{Fl+a} a^{-r} \lambda^{Fl+a} \sum_{s=0}^{\infty} \binom{-r}{s} \left(\frac{Fl}{a}\right)^s \\ &= \sum_{m=0}^{\infty} \binom{-r}{m} a^{-r} (-1)^a \lambda^a \left(\frac{F}{a}\right)^m \sum_{l=0}^{n-1} (-1)^l l^m \lambda^{Fl} \\ &= - \sum_{m=0}^{\infty} \binom{-r}{m} a^{-r} (-1)^a \lambda^a \left(\frac{F}{a}\right)^m \frac{1}{2} \{ \lambda^{Fn} \sum_{l=0}^{m-1} \binom{m}{l} E_l(\lambda^F) n^{m-l} \\ &\quad + (\lambda^{Fn} - 1) E_m(\lambda^F) \}. \end{aligned} \quad (27)$$

Hence, we note that

$$\begin{aligned} &- \sum_{l=0}^{n-1} \frac{\lambda^{Fl+a} (-1)^{Fl+a}}{(Fl+a)^r} - \frac{(\lambda^{Fn} - 1)}{2} \sum_{m=0}^{\infty} \binom{-r}{m} a^{-r} (-1)^a \lambda^a \left(\frac{F}{a}\right)^m E_m(\lambda^F) \\ &= \frac{1}{2} \sum_{m=0}^{\infty} \binom{-r}{m} a^{-r} (-1)^a \lambda^a \left(\frac{F}{a}\right)^m \lambda^{Fm} \sum_{l=0}^{m-1} \binom{m}{l} n^{m-l} E_l(\lambda^F) \\ &= \lambda^{Fn} \frac{(-1)^a a^{-r} \lambda^a}{2} \sum_{s=0}^{\infty} \binom{-r}{s} \left(\frac{F}{a}\right)^s \left(\sum_{l=0}^{s-1} \binom{s}{l} n^{s-l} E_l(\lambda^F) \right) \\ &= \lambda^{Fn} \sum_{s=0}^{\infty} \binom{-r}{s} w^{-r}(a) \left(\frac{F}{a}\right)^s \frac{(-1)^a \lambda^a}{2} \langle a \rangle^{-r} \sum_{l=0}^{s-1} \binom{s}{l} n^{s-l} E_l(\lambda^F) \\ &= \lambda^{Fn} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{-r}{k+l} w^{-r}(a) \left(\frac{a}{F}\right)^{-k-l} n^k \frac{(-1)^a \lambda^a}{2} \langle a \rangle^{-r} E_l(\lambda^F) \binom{k+l}{l} \\ &= \lambda^{Fn} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{r}{r+k} \binom{-r-1}{k} \binom{-r-k}{l} a^{-r} \left(\frac{a}{F}\right)^{-k-l} n^k \frac{(-1)^a}{2} \lambda^a E_l(\lambda^F) \\ &= \lambda^{Fn} \sum_{k=0}^{\infty} \frac{r}{r+k} \binom{-r-1}{k} w^{-k-r}(a) (nF)^k \frac{(-1)^a}{2} \lambda^a \langle a \rangle^{-k-r} \sum_{l=0}^{\infty} \binom{-r-k}{l} \left(\frac{F}{a}\right)^l E_l(\lambda^F) \\ &= \lambda^{Fn} \sum_{k=0}^{\infty} \frac{r}{r+k} \binom{-r-1}{k} w^{-k-r}(a) (nF)^k H_{\lambda,p}(r+k, a|F). \end{aligned} \quad (28)$$

For $F = p$, $r \in \mathbb{N}$, $n(=\text{even}) \in \mathbb{N}$, we have

$$2 \sum_{\substack{j=1 \\ (j,p)=1}}^{np} \frac{(-1)^j \lambda^j}{j^r} = 2 \sum_{a=1}^{p-1} \sum_{l=0}^{n-1} \frac{(-1)^{a+pl} \lambda^{a+pl}}{(a+pl)^r}. \quad (29)$$

Now, we set

$$B^{(r)}(a, F) = \frac{1}{2} \sum_{m=0}^{\infty} \binom{-r}{m} a^{-r} (-1)^a \lambda^a \left(\frac{F}{a} \right)^m E_m(\lambda^F). \quad (30)$$

From (27), (28), (29) and (30), we derive

$$\begin{aligned} & 2 \sum_{\substack{j=1 \\ (j,p)=1}}^{np} \frac{(-1)^j \lambda^j}{j^r} \\ &= 2 \sum_{a=1}^{p-1} \sum_{l=0}^{n-1} \frac{(-1)^{a+pl} \lambda^{a+pl}}{(a+pl)^r} \\ &= -2 \sum_{a=1}^{p-1} \left\{ \lambda^{pn} \sum_{k=0}^{\infty} \frac{r}{r+k} \binom{-r-1}{k} w^{-k-r}(a) (np)^k H_{\lambda,p}(r+k, a|F) \right. \\ & \quad \left. + (\lambda^{pn} - 1) B^{(r)}(a, p) \right\} \\ &= - \sum_{k=0}^{\infty} \frac{r}{r+k} \binom{-r-1}{k} \lambda^{pn} (pn)^k l_{\lambda,p}(r+k, w^{-k-r}) - 2(\lambda^{pn} - 1) \sum_{a=1}^{p-1} B^{(r)}(a, p). \end{aligned}$$

Therefore we obtain the following :

Theorem 10. *Let p be an odd prime and let n be an even positive integer. If $r \in \mathbb{N}$, then we have*

$$\begin{aligned} 2 \sum_{\substack{j=1 \\ (j,p)=1}}^{np} \frac{(-1)^j \lambda^j}{j^r} &= - \sum_{k=0}^{\infty} \frac{r}{r+k} \binom{-r-1}{k} \lambda^{pn} (pn)^k l_{\lambda,p}(r+k, w^{-k-r}) \\ &\quad - 2(\lambda^{pn} - 1) \sum_{a=1}^{p-1} B^{(r)}(a, p). \end{aligned}$$

Corollary 11. *Let $\lambda = 1$. Then we see [21] that*

$$2 \sum_{\substack{j=1 \\ (j,p)=1}}^{np} \frac{(-1)^j}{j^r} = - \sum_{k=0}^{\infty} \frac{r}{r+k} \binom{-r-1}{k} (pn)^k l_{1,p}(r+k, w^{-k-r}).$$

Remark. Luo-Srivastava have studied λ -Bernoulli numbers and polynomials [17, 18]. In [17, 18], λ -Bernoulli numbers and polynomials are called by Apostol-Bernoulli numbers and polynomials. Indeed, these numbers are Euler numbers [2,3,6]. In [6], we have studied these numbers and polynomials by using the “bosonic” expression of p -adic q -integral on \mathbb{Z}_p . That is, $\frac{\log \lambda + t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}$.

However, this generating function is not simple. So, we modified these generating function of $B_n(\lambda)$ in the locally constant space.

If $\lambda \in T_p = \bigcup_{n \geq 0} \{\zeta | \zeta^{p^n} = 1\}$, then we have

$$\frac{\log \lambda + t}{\lambda e^t - 1} = \frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}, \quad \text{cf. [2, 3, 4, 6, 19].}$$

Recently, Simsek, Jang-Pak-Rim also studied for the properties of λ -Bernoulli numbers and polynomials (see [2,3,19]). In [16], M. Schork also studied the representation of the q -fermionic commutation relations and limit $q = 1$.

§5. Applications of p -adic q -integrals at $q = -1$ (or $q = 1$)

In this section we consider p -adic q -integral at $q = -1$ (or $q = 1$) involving trigonometric functions. By Eq.(7), we see that $I_{-1}(f_1) + I_{-1}(f) = 2f(0)$, for $f \in C^{(1)}(\mathbb{Z}_p)$. If we take $f(x) = \cos ax$ (or $f(x) = \sin ax$), then we have

$$\begin{aligned} 0 &= \int_{\mathbb{Z}_p} \sin a(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) \\ (31) \quad &= (\cos a + 1) \int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x) + \sin a \int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x), \end{aligned}$$

and

$$\begin{aligned} 2 &= \int_{\mathbb{Z}_p} \cos a(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x) \\ (32) \quad &= (\cos a + 1) \int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x) - \sin a \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x). \end{aligned}$$

Thus, we note that

$$(33) \quad \begin{aligned} \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) &= -\frac{\sin a}{\cos a + 1} \int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x), \\ 2(\cos a + 1) &= (\cos a + 1)^2 \int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x) + \sin^2 a \int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x). \end{aligned}$$

From (31), (32) and (33), we derive

$$(34) \quad \int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x) = 1, \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) = -\frac{\sin a}{\cos a + 1}.$$

Therefore we obtain the below:

Proposition 12. *Let $a \in \mathbb{Z}_p$. Then we have*

$$\int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x) = 1, \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) = -\frac{\sin a}{\cos a + 1} = -\tan \frac{a}{2}.$$

By using Taylor expansion for $\sin ax$ at $x = 0$, we easily see that

$$(35) \quad \sin ax = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} a^{2n+1} x^{2n+1}.$$

From (9-1), (34) and (35), we derive

$$(36) \quad -\tan \frac{a}{2} = \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)!} \int_{\mathbb{Z}_p} x^{2n+1} d\mu_{-1}(x).$$

Theorem 13. *Let $\lim_{q \rightarrow -1} \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x)$. Then we have*

$$\tan \frac{a}{2} = - \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} a^{2n+1}}{(2n+1)!} E_{2n+1},$$

where E_n are the n -th Euler numbers.

Let $f_1(x) = f(x+1)$ be translation in $C^1(\mathbb{Z}_p)$ and let $f'(0) = \frac{d}{dx} f(x)|_{x=0}$. Then we note that

$$(37) \quad I_1(f) = \lim_{q \rightarrow -1} \int_{\mathbb{Z}_p} f(x) d\mu_q(x) \implies I_1(f_1) - I_1(f) = f'(0).$$

By (37), we easily see that

$$\begin{aligned} 0 &= \int_{\mathbb{Z}_p} \cos a(x+1) d\mu_1(x) - \int_{\mathbb{Z}_p} \cos ax d\mu_1(x) \\ &= (\cos a - 1) \int_{\mathbb{Z}_p} \cos ax d\mu_1(x) - \sin a \int_{\mathbb{Z}_p} \sin ax d\mu_1(x). \end{aligned}$$

Thus, we have

$$(38) \quad \int_{\mathbb{Z}_p} \sin ax d\mu_1(x) = -\frac{a}{2}.$$

It is easy to see that

$$\begin{aligned} (39) \quad a &= \int_{\mathbb{Z}_p} \sin a(x+1) d\mu_1(x) - \int_{\mathbb{Z}_p} \sin ax d\mu_1(x) \\ &= (\cos a - 1) \int_{\mathbb{Z}_p} \sin ax d\mu_1(x) + \sin a \int_{\mathbb{Z}_p} \cos ax d\mu_1(x). \end{aligned}$$

From (38) and (39), we derive

$$(40) \quad \int_{\mathbb{Z}_p} \cos ax d\mu_1(x) = \frac{a \sin a}{2 - 2 \cos a} = \frac{a}{2} \cot \frac{a}{2}.$$

By using Taylor expansion, we easily see that

$$\cos ax = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{(2n)!} x^2.$$

Thus, we have

$$\frac{a}{2} \cot a = \int_{\mathbb{Z}_p} \cos ax d\mu_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n B_{2n}}{(2n)!} a^{2n},$$

where B_n are the n -th Bernoulli numbers.

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